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On the topological orbit equivalence in a class of substitution minimal systems

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In this note, a partial answer to the problem to characterize the topological orbit equivalence class of substitution minimal systems. The characterization is given in terms of the Perron-Frobenius eigenvalue of a matrix associated with a substitution.

1. Topological orbit equivalence in Cantor systems

A topological dynamical system (X,ϕ) is called a Cantor system if X is a Cantor set and ϕ is a minimal homeomorphism on X.

Definition 1.1. Let (X,ϕ) be a Cantor system. We put

$$\begin{aligned}\tilde{K}^0(\phi) &= C(X,\mathbb{Z})/Z_\phi\\ \tilde{K}^0_+(\phi) &= C(X,\mathbb{Z}_+)/Z_\phi\\ \tilde{u}_\phi &= [1]\end{aligned}$$

where $C(X,\mathbb{Z})$ is the abelian group of continuous functions on X with integer values, [1] is the equivalence class of the constant function 1 by the subgroup Z_ϕ and

$$Z_\phi = \{f\in C(X,\mathbb{Z}) | \int_X f d\mu = 0 \text{ for every } \phi\text{-inv. prob. meas. on } X\}.$$

Definition 1.2. Let (X_i,ϕ_i) be Cantor systems for $i=1,2$. ϕ_1 and ϕ_2 are said to be topologically orbit equivalent if there exists a homeomorphism $F: X_1\to X_2$ such that $F(\mathrm{Orb}_{\phi_1}(x)) = \mathrm{Orb}_{\phi_2}(F(x))$ for every $x\in X_1$ where $\mathrm{Orb}_{\phi_i}(y)$ is the orbit of y by ϕ_i.

Theorem 1.3 ([GPS]). *The triple $(\tilde{K}^0(\phi),\tilde{K}^0_+(\phi),\tilde{u}_\phi)$ is a complete invariant of the topological orbit equivalence in the class of Cantor systems.*

Put

$$\begin{aligned}&X = \prod_{i=1}^{\infty}\{0,1,\dots,n_i\}, \quad n_i\geq 2,\\ &\phi: X\to X, \ \text{ the addition of } (1,0,0,\dots) \text{ with carries.}\end{aligned}$$

Then, (X,ϕ) is a Cantor systems and called the odometer system with base $(n_1,n_2,n_3,\dots)$. The invariant $\tilde{K}^0$ of ϕ is the group $\{l/m | l\in\mathbb{Z},\ m \text{ divides some } \prod_{i=1}^k n_i\}$ and we denote the group of this form by $\mathbb{Z}_{(q)}$ where $q=\prod_{i=1}^\infty n_i$ as a formal product. The invariant $\tilde{K}^0$ of ϕ is $(\mathbb{Z}_{(q)}, \mathbb{Z}_{(q)}\cap\mathbb{R}_+, 1)$ as the triple.

2. Definition of substitution systems

Let A be an alphabet, i.e. a finite set, and A^+ be the set of words on A.

Definition 2.1. A map $\sigma : A \to A^+$ is called a substitution on A.

Let σ be a substitution on A. A substitution σ is naturally extended on A^+ and $A^{\mathbb{Z}}$. We put $\mathcal{L}(\sigma) = \{u \in A^+ | u \text{ occurs in some } \sigma^k(a), k \geq 1, a \in A\}$ and denote by $M(\sigma)$ the $A \times A$ matrix whose (a, b)-entry is the number of occurrences of b in $\sigma(a)$ and call it the composition matrix of σ. A substitution σ is said to be of constant length if the length of $\sigma(a)$ does not depend on the choice of a and to be primitive if there exists an integer $k \geq 1$ such that for every $a, b \in A$, a occurs in $\sigma^k(b)$, equivalently $M(\sigma)$ is a primitive matrix.

Remark 2.2. As the alphabet A is a finite set, there exist an integer $k \geq 1$ and letters a, b such that

1. a is a prefix of $\sigma^k(a)$;
2. b is a suffix of $\sigma^k(b)$;
3. $ba \in \mathcal{L}(\sigma)$.

Then $x = \lim_{n\to\infty} \sigma^{kn}(b).\sigma^{kn}(a)$ converges in $A^{\mathbb{Z}}$ where the dot means the separation between the -1-st coordinate and the 0-th one.

Remark 2.3. We always assume that every substitution σ in this note satisfies the following conditions:

1. there exists a letter a such that $\lim_{n\to\infty} \sigma^n(a) = \infty$;
2. a point x given as above is aperiodic.

Let T be a bilateral shift on $A^{\mathbb{Z}}$ and X_σ be the closure of the orbit of x by T. Put $T_\sigma = T|_{X_\sigma}$.

Definition 2.4. The substitution system arising from a substitution σ is (X_σ, T_σ).

Proposition 2.5 ([Qu]). *If a substitution σ is primitive, then T_σ is uniquely ergodic and minimal.*

We always assume that every substitution in this note is primitive.

3. The invariant $\tilde{K}^0(T_\sigma)$.

Definition 3.1. A substitution σ is said to be proper if there exist an integer $k \geq 1$ and letters a, b such that for every letter c, a is a prefix of $\sigma^k(c)$ and b is a suffix of $\sigma^k(c)$.

Remark 3.2 ([DHS]). A proper substitution is not a special one from the view point of dynamical systems because for every substitution σ there exists a proper substitution ζ such T_ζ is topologically conjugate to T_σ.

We first consider the case where a substitution σ is proper.

Definition 3.3. We put

$$K^0(T_\sigma) = \varinjlim(M(\sigma) : \mathbb{Z}^s \to \mathbb{Z}^s) \quad \text{where } s = |A|,$$
$$K^0_+(T_\sigma) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^s_+),$$
$$u_{T_\sigma} = {}^t(1, \dots, 1),$$

where φ_n is a natural homomorphism, which satisfies that $\varphi_n = \varphi_{n+1}M(\sigma)$ and $K^0(T_\sigma) = \bigcup_{n=1}^{\infty} \varphi_n(\mathbb{Z}^s)$. Define $p_\sigma : K^0(T_\sigma) \to \mathbb{R}$ by $p_\sigma(\varphi_n(a)) = \lambda^{-(n-1)}\alpha(a)$ for $a \in \mathbb{Z}^s$ where λ is the Perron- Frobenius eigenvalue of $M(\sigma)$ and α is the left eigenvector corresponding to λ such that $\sum_i \alpha_i = 1$.

Theorem 3.4 (From a result of [DHS]). *The invariant* $(\tilde{K}^0(T_\sigma), \tilde{K}^0_+(T_\sigma), \tilde{u}_{T_\sigma})$ *defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system* (X_σ, T_σ) *is* $(K^0(T_\sigma)/\ker(p_\sigma), K^0_+(T_\sigma)/\ker(p_\sigma), p_\sigma(u_{T_\sigma})) = (\mathrm{Im}(p_\sigma), \mathrm{Im}(p_\sigma) \cap \mathbb{R}_+, 1)$.

Therefore, if λ is rational,i.e. integral, then $\tilde{K}^0(T_\sigma) = \mathbb{Z}_{(d\cdot\lambda^\infty)}$ for some integer $d \geq 1$.

Next, we consider the case where a substitution σ is not proper.

Definition 3.5. A word $u \in \mathcal{L}(\sigma)$ is a return word to ba, where a and b are letters, if

1. a is a prefix of u.
2. b is a suffix of u.
3. $bua \in \mathcal{L}(\sigma)$.
4. ba occurs in bua only twice.

Remark 3.6. The number of return words is finite because of the minimality of T_σ. The length of a return word u to ba is the first return time to the cylider set $[b.a]$ of the points in the cylider set $[b.ua]$ where $[u.v] = \{y \in X_\sigma | y_{[-|u|,|v|)} = uv\}$ for words u, v.

Fix an integer $k \geq 1$ and letters a, b such that the conditions of Remark 2.2 hold. Put $W = \{w_1, \dots, w_r\}$ indexed in order of occurence without multiplicities in $x_{[0,+\infty)}$. Define a substitution τ on the alphabet $R = \{1, \dots, r\}$ by

$$\tau(i) = i_1 \dots i_l \quad \text{if} \quad \sigma^k(w_i) = w_{i_1} \dots w_{i_l}.$$

Proposition 3.7 ([DHS]). *The substitution τ defined as above is primitive and proper. The substitution system arising from τ is topologically conjugate to the induced transformation on $[b.a]$ by T_σ.*

Definition 3.8. We put

$$K^0(T_\sigma) = \varinjlim(M(\tau) : \mathbb{Z}^r \to \mathbb{Z}^r),$$
$$K^0_+(T_\sigma) = \bigcup_{n=1}^{\infty} \psi_n(\mathbb{Z}^r_+),$$
$$u_{T_\sigma} = {}^t(|w_1|, \dots, |w_r|),$$

where ψ_n is a natural homomorphism as in Definition 3.3. Define $p_\sigma : K^0(T_\sigma) \to \mathbb{R}$ by $p_\sigma(\psi_n(a)) = \mu^{-(n-1)}\beta(a)$, $a \in \mathbb{Z}^r$, where μ is the Perron-Frobenius eigenvalue of $M(\tau)$ and β is the left eigenvector corresponding to β such that $\sum_i \beta_i |w_i| = 1$.

Theorem 3.9 (From a result of [DHS]). *The invariant* $(\tilde{K}^0(T_\sigma), \tilde{K}^0_+(T_\sigma), \tilde{u}_{T_\sigma})$ *defined in Definition 1.1 of the topologically orbit equivalence for a substitution minimal system* (X_σ, T_σ) *is* $(K^0(T_\sigma)/\ker(p_\sigma), K^0_+(T_\sigma)/\ker(p_\sigma), p_\sigma(u_{T_\sigma})) = (\mathrm{Im}(p_\sigma), \mathrm{Im}(p_\sigma) \cap \mathbb{R}_+, 1)$.

Therefore, if μ is integral, then $\tilde{K}^0(T_\sigma) = \mathbb{Z}_{(d' \cdot \mu^\infty)}$ for some integer $d' \geq 1$.

Remark 3.10. Given a substitution σ, there exist an infinite graph and a partial order on the edge set of the graph which induces a minimal homeomorphism on the infinite path space which is topologically conjugate to T_σ. If σ is proper, then the connection rule between vertices in the corresponding graph is given by $M(\sigma)$. If σ is not proper, then the connection rule is given by $M(\tau)$. This is the reason why the way to compute the invariant $\tilde{K}^0(T_\sigma)$ is different between the case where σ is proper and the case where σ is not proper. See [DHS] for more details.

Theorem 3.11 ([Yu]). *Let* σ *be a substitution whose* $M(\sigma)$ *has an integral Perron-Frobenius eigenvalue* λ. *Then, the substitution system arising from the substitution* σ *is topologically orbit equivalent to the odometer system with base* $(d, \lambda, \lambda, \dots)$ *(called a stationary odometer system) for some integer* $d \geq 1$. *In particular, every substitution system arising from a substitution of constant length is topologically orbit equivalent to a stationary odometer system.*

Key lemma for the proof is the following.

Lemma 3.12. $\mu = \lambda^k$.

Proof. Let S be an $R \times A$ matrix whose (a, i)-entry is the number of occurrences of a in w_i. Then $SM(\sigma)^k = M(\tau)S$. Therefore, $\mu = \lambda^k$ because of the Perron-Frobenius Theorem. □

Remark 3.13. When σ is proper, $d = \sum_i \alpha_i$ where $\alpha = (\alpha_1, \dots, \alpha_s)$ is the left Perron-Frobenius eigenvector of $M(\sigma)$ such that every α_i is integral and $(\alpha_i, \alpha_j) = 1$ if $i \neq j$. When σ is not proper, $d' = \sum_i \beta_i |w_i|$ where $\beta = (\beta_1, \dots, \beta_r)$ is the left Perron-Frobenius eigenvector of $M(\tau)$ such that each β_i is integral and $(\beta_i, \beta_j) = 1$ if $i \neq j$.

The converse of Theorem 3.11:

Theorem 3.14 ([Yu]). *Let* (X, ϕ) *be an arbitrary stationary odometer system and its base be* $(d, \lambda, \lambda, \dots)$. *Then, there exists a proper and primitive substitution* σ *of constant length such that* T_σ *is topologically orbit equivalent to* ϕ.

Proof. We may assume that $d > 1$ and $\lambda > 1$. It is enough to show that there exists a proper and primitive substitution σ of constant length λ^n on the alphabet $\{1, \dots, d\}$ for some integer $n \geq 1$. Take $n \geq 1$ such that $\lambda^n > 3 \vee d$. Put $v = {}^t(\lambda^m, \lambda^m, \dots, \lambda^m, (\lambda^m - d + 1)\lambda^m)$. Let M be the integral $d \times d$ matrix whose (i, j)-entry is the $\kappa^{j-1}(i)$-th entry of v for $1 \leq i, j \leq d$ where κ is the permutation on $\{1, 2, \dots, d\}$ defined by $\kappa(d) = 1$ and $\kappa(i) = i + 1$ if $1 \leq i < d$. We can find a proper and primitive substitution σ such that $M(\sigma) = M$. □

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